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Fuzzy-π-Topology and Fuzzy -π- Continuity Via Fuzzy-Ideal-Topological-Spaces

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Abstract:

This paper introduce the concept of fuzzy-π-open sets and fuzzy-π-topology, along with the definitions of fuzzy-π_i-continuity and fuzzy-π_{ii}-continuity using the concept of fuzzy ideal topological spaces. We also explore the relationships among fuzzy-π-open sets and other related concepts such as fuzzy-semi-open, fuzz-alpha-open, fuzzy-regular-open and fuzzy-preopen. Additionally, we investigate the properties of fuzzy-π-topological space and their relationships with other type of continuities.

الطوبولوجيا الضبابية والاستمرارية الضبابية عبر الفضاءات الطوبولوجية المثالية الضبابية

الخلاصة

يقدم هذا البحث مفهوم المجموعات المفتوحة الضبابية والطوبولوجيا الضبابية، بالإضافة إلى تعريفات الاستمرارية الضبابية والاستمرارية الضبابية باستخدام مفهوم الفضاءات الطوبولوجية المثالية الضبابية . نحن نستكشف أيضًا العلاقات بين المجموعات المفتوحة الغامضة والمفاهيم الأخرى ذات الصلة مثل المجموعة شبه المفتوحة الغامضة، والفتحة ألفا الغامضة، والفتحة العادية الغامضة، والفتحة الغامضة المسبقة. بالإضافة إلى ذلك، قمنا بدراسة خصائص الفضاء الطوبولوجي الغامض وعلاقاتها مع أنواع أخرى من الاستمراريات.

1. INTRODUCTION

Kuratowski [10] and Vaidyanathaswamy [8] introduced the notion of the ideal and ideal topological spaces also they define the local function. Jankovic and Hamlet [7] studied the ideal in general topology. The concept of fuzzy sets was introduced by Zadeh in 1965 [11]. In 1968 Chang present the notion of fuzzy topology and introduce concept of fuzzy interior, fuzzy closure and fuzzy continuity [5]. Since the various attributes of general topology was studied in fuzzy sense by several Researchers in this field. K.K. Azad [9] and Bin Shahan A.S [4] introduce concept of fuzzy semi-continuous and fuzzy pre-continuous functions, also M. K. Singal and N. Rajvanshi [10] introduced notion of fuzzy alpha-continuous function. In 1997 D. Sarkar [8] introduce notions of fuzzy ideal theory and fuzzy local function in fuzzy ideal topological spaces. M.K. Chakraborty and T.M.G. Ahsanullah [12] and Pu Pao-ming and Liu Ying-ming they define quasi-coincident sets and quasi-neighborhood of fuzzy sets and points. In section three we introduce fuzzy-π-open sets, fuzzy-π-topology with some relationships with other sets. Section four introduce fuzzy-π_i-continuity, fuzzy-π_{ii}-continuity and fuzzy-π_{ii}-homeomorphism with some relationships and properties.

2. PRELIMINARIES

Consider a nonempty set Ӽ. We define a fuzzy set S of Ӽ using its membership function μ_S: Ӽ $\rightarrow$ [0, 1]. The value μ_S(x) represent the membership degree of x in S, for all x $\in$ Ӽ. This relationship between S and its membership degree represented by the pairs S = {(x, μ_S(x)): x $\in$ Ӽ} [11]. The class T_f of fuzzy sets in Ӽ is called fuzzy topology for Ӽ if, 0_x, 1_x $\in$ T_f, T_f closed under arbitrary union and finite intersection [5]. The fuzzy-topological space briefly (fts) symbolized by (Ӽ, T_f) with elements of T_f being termed fuzzy-open sets and the fuzzy set F is fuzzy-closed if its complement is fuzzy-open. The fuzzy point is a fuzzy set takes a non-zero value $\alpha \in (0, 1]$ at a single element x in Ӽ, and denoted by x_α with x being its Support [15]. Let A and B be a fuzzy subsets of fuzzy topological space (Ӽ, T_f) then we say that A is fuzzy quasi-coincident with B if, $\exists$ x $\in$ Ӽ such that A(x) + B(x) $>$ 1 and written as AqB [15]. A fuzzy set U in fts (X, T_f) is said to be a quasi-neighborhood (in short, q-nbd) of a fuzzy point x_α if and only if there exist a fuzzy open set V $\subseteq$ U such thatx_αqV. We will denote the set of all q-nbd of x_α in (X, T_f) by N(x_α) [12, 15].
A class I_f of fuzzy subsets of Ӽ is called a fuzzy ideal on Ӽ [8] iff
(i) If $\mu \in I_f$ and v $\subseteq \mu$, then v $\in I_f$ [heredity]
(ii) If μ, v $\in I_f$, then $\mu \cup$ v [finite additivity].
Let (Ӽ, T_f) be fts and I_f be a fuzzy ideal on Ӽ, then the triple (Ӽ, T_f, I_f) is called fuzzy ideal topological space. And the fuzzy local function $S^*(I_f, T_f)$ [briefly S^*] of a fuzzy subset S of Ӽ is the union of all fuzzy points x_α such that if G $\in$ N(x_α) and E $\in I_f$ then there is at least one y $\in$ Ӽ for which G(y) + S(y) - 1 > E(y). Therefore any $x_\alpha \notin S^*(I_f, T_f)$ (for any $x_\alpha \notin$ S) implies to x_α is not contained in S, i.e. $\alpha >$ S(x)) implies there is at least one G $\in$ N(x_α) such that for every y $\in$ Ӽ, G(y) + S(y) - 1 $\leq$ E(y) for some E $\in I_f$ [8]. Let us define $cl^*(S) = S \cup S^*$ for any fuzzy subset S of Ӽ. It is clear that cl^* is a fuzzy closure operator. Let $T^*(I_f)$ be the fuzzy topology generated by cl^*, $T^*(I_f) = \{S;\ cl^*(S^c) = S^c\}$ [8]. Now if I_f = {0_x} then $cl^*(S) = S \cup S^* = S \cup cl(S) = cl(S)$, for every S $\in$ P(Ӽ), so $T^*(\{0_x\}) = T_f$. Again if I_f = P(Ӽ) the $cl^*(S) = S$, because $S^* = 0_x$, for every S $\in$ P(Ӽ). So, T^*(P(Ӽ)) is the fuzzy discrete topology on Ӽ. Since {0_x} and P(Ӽ) are the two extreme fuzzy ideals on Ӽ, therefore for any fuzzy ideal I_f on Ӽ, we have $T^*(\{0_x\}) \subseteq T^*(I_f) \subseteq T^*$(P(Ӽ)). In particular for any two fuzzy ideals J_f and I_f on Ӽ such that $J_f \subseteq I_f$ we have $T^*(I_f) \subseteq T^*(J_f)$. Also every element of fuzzy ideal is T^*-closed. The fuzzy ideal I_f on a fuzzy space (Ӽ, T_f) is called codense ideal if $I_f \cap T_f$ = {0_x} [1]. A fuzzy operator ψ_{Tf}: P(Ӽ) $\longrightarrow T_f$ for every fuzzy subset S of Ӽ of fuzzy ideal topological space (Ӽ, T_f, I_f), defined as $\psi_{Tf}(S) = \{x_\alpha: \exists$ G $\in$ N(x_α) such that G - S $\in$ I} or $\psi_{Tf}(S) = 1_x - (1_x - S)^*$ and $T^*(I_f)$ = {S: S $\subseteq \psi_{Tf}$(S)}. The fuzzy subset S of a fuzzy ideal topological space (Ӽ, T_f, I_f) is called fuzzy T^*–closed if $S^* \leq S$ [8], fuzzy $*$–dense in itself if $S \leq S^*$[8]. And fuzzy-$*$-perfect if $S = S^*$.

<u>**Definition 2.2**</u>:-[11] Given C and D be two fuzzy subset of Ӽ defined as C = {(x, μ_C(x)): x $\in$ Ӽ} and D = {(x, μ_D(x)): x $\in$ Ӽ}, then we establish the following definitions;
1- C $\subseteq$ D iff $\mu_C(x) \leq \mu_D(x)$
2- C = D iff C $\subseteq$ D and D $\subseteq$ C
3- G = C $\cup$ D iff $\mu_G(x) = max\ \{\mu_C(x), \mu_D(x)\}$
4- F = C $\cap$ D iff $\mu_F(x) = min\ \{\mu_C(x), \mu_D(x)\}$
5- E = D^c iff $\mu_E(x) = 1 - \mu_D(x)$
6- 1_x = Ӽ = {(x, 1): x $\in$ Ӽ} and 0_x = $\emptyset$ = {(x, 0): x $\in$ Ӽ}.

Definition 2.3:-[11] Let $\{A_j\}$ $j \in J$ be a family of fuzzy subsets of Ҳ, then we define

1- $G = \cup_{j\in J} D_j$ iff $\mu_G(x) = sup\ \{\mu_{Dj}(x)\}$

2- $F = \cap_{j\in J} D_j$ iff $\mu_F(x) = inf\ \{\mu_{Dj}(x)\}$.

Theorem 3.4:-[8] Let I_f and J_f be two fuzzy ideals on Ҳ in fts (Ҳ, T_f), Then for any fuzzy sets A, B of Ҳ,

(i) $A \subseteq B$, then $A^* (I_f, T_f) \subseteq B^*(I_f, T_f)$

(ii) If $I_f \subseteq J_f$, then $A^* (J_f, T_f) \subseteq A^*(I_f, T_f)$

(iii) $A^* = cl(A^*) \subseteq cl(A)$

(iv) $(A^*)^* \subseteq A^*$,

(v) $(A \cup B)^* = A^* \cup B^*$,

(vi) If $E \in I$, then $(A \cup E)^* = A^*$.

Definition 2.1:-[2] Given f: (Ҳ, T_f) → (Y, σ_f) and B be a fuzzy subset of Y with membership function $\mu_B(y)$. Then inverse image of B is a fuzzy subset of Ҳ whose membership function is defined by $\mu_{f^{-1}_{(B)}}(x) = \mu_B(f(x))\ \forall\ x \in$ Ҳ. And if A is a fuzzy subset of Ҳ with membership function $\mu_A(x)$. The image of A is a fuzzy subset of Y whose membership function is given by

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}_{(y)}} \{\mu_A(x)\} & \text{if } \mu_{f^{-1}_{(y)}}(x) \text{ is nonempty} \\ 0, & \text{otherwise} \end{cases}$$

For each y belong to Y, where $\mu_{f^{-1}_{(y)}}(x) = \mu_{f(x)}(y) = \mu_y(y)$

3. Fuzzy-π-Open Sets.

Definition 3.1:- A fuzzy subset S of Ҳ in fuzzy ideal topological space (Ҳ, T_f, I_f) termed fuzzy-π-open if there exist fuzzy-T^*-open set G such that $G \subseteq S \cap \psi_{T_f}(S)$. i. e ($\mu_G(x) \leq \min\ \{\mu_S(x), \mu_{\psi_{T_f}(S)}(x)\}$ for each x in Ҳ. [$G = 0_x$ iff $S = 0_x$]. The class of all fuzzy-π-open in Ҳ symbolized by π_f(Ҳ)

Example 3.2:- Given Ҳ = $\{s_1, s_2\}$ and the fuzzy subsets of Ҳ are $A = \{(s_1, 0.5), (s_2, 0.7)\}$, $B = \{(s_1, 0.8), (s_2, 0.8)\}$ and $D = \{(s_1, 0.8), (s_2, 0.9)\}$. Let $T_f = \{1_x, 0_x, A, B\}$ and $I_f = \{0_x\}$. $T_f^* = \{1_x, 0_x, A, B\}$ and π_f(Ҳ) $= \{1_x, 0_x, A, B, D\}$.

Example 3.3:- Let Ҳ = $\{s_1, s_2\}$ and the fuzzy subsets of Ҳ are $A = \{(s_1, 0.5), (s_2, 0.7)\}$, $B = \{(s_1, 0.8), (s_2, 0.8)\}$, $C = \{(s_1, 0.7), (s_2, 0.3\}$ and $D = \{(s_1, 0.6), (s_2, 0.9)\}$. Let $T_f = \{1_x, 0_x, A, B\}$ and $I_f = \{0_x\}$. $T_f^* = \{1_x, 0_x, A, B\}$. And π_f(Ҳ) $= \{1_x, 0_x, A, B, D\}$. Since C is not fuzzy- π-open.

Lemma 3.4:- Let, A, B be two fuzzy subsets of Ҳ and $A \subseteq B$, then $\psi_{Tf}(A) \subseteq \psi_{Tf}(B)$.

Proof:- Suppose $A \subseteq B$, then $\mu_A(x) \leq \mu_B(x)$ for every $x \in$ Ҳ, that is $(1_x - B(x)) \leq (1_x - A(x))$ for every $x \in$ Ҳ. Therefore $1_x - (1_x - A) \leq 1_x - (1_x - B)$. Thus from definition of ψ_{Tf} we have $\psi_{Tf}(A) \subseteq \psi_{Tf}(B)$.

Theorem 3.5:- The class of all fuzzy-π-open sets with respect to T_f in (Ҳ, T_f, I_f) space is a fuzzy topology on Ҳ.

Proof:- 1- It is clear that 0_x and 1_x are fuzzy-π-open sets by [definition 3.1].

2- Suppose that $\{S_i\}$ be an arbitrary family of fuzzy-π-open sets such that $S_i = \{<x, \mu_{Si}(x)>, x \in$ Ҳ$\}$, and let $\{G_i\}$ be a family of fuzzy-T^*-open sets in Ҳ such that $G_i = \{<x, \mu_{Gi}(x)>, x \in$ Ҳ$\}$ and $G_i \subseteq S_i \cap \psi_{Tf}(S_i)$ for each i. Therefore

$\mu_{Gi}(x) \leq \mu_{Si}(x)$ for each i and x (1)

$\mu_{Gi}(x) \leq \mu_{\psi_{Tf(Si)}}(x)$ for each i and x (2)

That is

$\vee_i \mu_{Gi}(x) \leq \vee_i \mu_{Si}(x)$ (3)

$\vee_i \mu_{Gi}(x) \leq \vee_i \mu_{\psi_{Tf(Si)}}(x)$ (4)

From (3) and (4) we get $\cup G_i \subseteq \cup S_i \cap (\cup\ \psi_{Tf}(S_i))$ (5)

Since $\{G_i\}$ is a family of fuzzy-T^*-open sets, therefore $\cup G_i$ is fuzzy-T^*-open set, thus $\cup G_i \subseteq \psi_{Tf}(\cup G_i)$.

And from (5) we have $\cup G_i \subseteq \cup S_i$ (6)

From (6) and [**Lemma 3.4]** we get $\psi_{Tf}(\cup G_i) \subseteq \psi_{Tf}(\cup S_i)$ (7)

Since $\cup G_i \subseteq \psi_{Tf}(\cup G_i)$ and from (6) and (7) we have $\cup G_i \subseteq \cup S_i \cup \psi_{Tf}(\cup S_i)$. Hence $\cup S_i$ is fuzzy-π-open

3- Let S_1 and S_2 be two fuzzy-π-open sets, then there exist $G_1, G_2 \in T^*(I_f)$ such that
$G_1 \subseteq S_1 \cap \psi_{Tf}(S)$ and $G_2 \subseteq S_2 \cap \psi_{Tf}(S_2)$

Therefore

$\mu_{G1}(x) \leq \mu_{S1}(x)$ and $\mu_{G2}(x) \leq \mu_{S2}(x)$ for each x (8)

$\mu_{G1}(x) \leq \mu_{\psi_{Tf(S1)}}(x)$ and $\mu_{G2}(x) \leq \mu_{\psi_{Tf(S2)}}(x)$ for each x (9)

Therefore

min $\{\mu_{G1}(x), \mu_{G2}(x)\} \leq$ min $\{\mu_{S1}(x), \mu_{S2}(x)\}$ (10)

min $\{\mu_{G1}(x), \mu_{G2}(x)\} \leq$ min $\{\mu_{\psi_{Tf(S1)}}(x), \mu_{\psi_{Tf(S2)}}(x)\}$ (11)

From (10) and (11) we get

$G_1 \cap G_i \subseteq (S_1 \cap S_2) \cap (\psi_{Tf}(S_1) \cap \psi_{Tf}(S_2))$ (12)

Since G_1 and G_2 are fuzzy-T^*-open sets, therefore $G_2 \cap G_2$ is fuzzy-T^*-open set,
thus $G_1 \cap G_2 \subseteq \psi_{Tf}(G_2 \cap G_i)$.

And from (12) we have $G_1 \cap G_2 \subseteq S_1 \cap S_2$ (13)

From (13) and [**Lemma 3.4]** we get $\psi_{Tf}(G_1 \cap G_2) \subseteq \psi_{Tf}(S_1 \cap S_2)$ (14)

Since $G_1 \cap G_2 \subseteq \psi_{Tf}(G_1 \cap G_2)$ and from (13) and (14) we get
$(G_1 \cap G_2) \subseteq (S_1 \cap S_i) \cup \psi_{Tf}(S_1 \cap S_i)$.

Therefore $S_2 \cap S_2$ is fuzzy-π-open set. Hence the class of all fuzzy-π-open sets is a fuzzy-topology on Ӽ.

Remark 3.6:- The class of all fuzzy-π-open sets with respect to T_f in (Ӽ, T_f, I_f) space is called fuzzy-π-topology and denoted by π_f, the pair (Ӽ, π_f) called fuzzy-π-topological space. An element of π_f is called fuzzy-π-open. The fuzzy subset F of Ӽ is called fuzzy-π-closed if its complement is fuzzy-π-open.

Definition 3.7:- Given S be a fuzzy subset of Ӽ in fuzzy-π-topological space (Ӽ, π_f), then fuzzy-π-interior of S briefly [π-int(S)] And fuzzy-π-closure of S briefly [π-cl(S)] is defined as; [π-int(S) = $\cup$ {G: G $\in \pi_f$,G $\subseteq$ S} and [π-cl(S) = $\cap$ {F: $F^c \in \pi_f$, S $\subseteq$ F} respectively.

Theorem 3.8:- In fuzzy ideal topological space (Ӽ, T_f, I_f) every fuzzy-T^*-open is fuzzy-π-open but not contrary.

Proof:- Suppose that S be a fuzzy-T^*-open set, therefore $S \subseteq \psi_{Tf}(S)$. That is $\mu_S(x) \leq \mu_{\psi_{Tf}(S)}(x)$ for every x $\in$ Ӽ. Since $\mu_S(x) = \mu_S(x)$ for every x $\in$ Ӽ, therefore $\mu_S(x) \leq$ min $\{\mu_S(x), \mu_{\psi_T(S)}(x)\}$ for every x in Ӽ. That is $S \subseteq S \cap \psi_{Tf}(S)$. Hence S is fuzzy-$\pi$-open.

Example 3.9 :- Let Ӽ = $\{s_1, s_2\}$ and the fuzzy subsets of Ӽ are, A = $\{(s_1, 0.5), (s_2, 0.3)\}$, and B = $\{(s_1, 0.7), (s_2, 0.6)\}$. Suppose $T_f = \{1_x, 0_x, A\}$ and $I_f = \{0_x\}$, then $T_f^* = \{1_x, 0_x, A\}$ and $\pi_f = \{1_x, 0_x, A, B\}$. Since B is fuzzy- π-open but not fuzzy-T^*-open.

Corollary 3.10:- Every Fuzzy open set is fuzzy-π-open but not contrary.

Proof:- Suppose S is fuzzy open set on Ӽ, since every fuzzy open is fuzzy-T^*-open. Therefore S is fuzzy- π-open by **[Theorem 3.8]**

Example 3.11 :- Let Ӽ = $\{s_1, s_2\}$ and the fuzzy subsets of Ӽ are A = $\{(s_1, 0.5), (s_2, 0.3)\}$, B = $\{(s_1, 0.3), (s_2, 0.3)\}$ and C = $\{(s_1, 0.7), (s_2, 0.6)\}$. Suppose $T_f = \{1_x, 0_x, A\}$ and $I_f = \{0_x\}$, then $T_f^* = \{1_x, 0_x, A\}$ and $\pi_f = \{1_x, 0_x, A, C\}$. Since C is fuzzy- π-open but not fuzzy-open.

Remark 3.12:- In fuzzy ideal topological space (Ӽ, T_f, I_f) fuzzy preopen sets and fuzzy-π-open are independent.

Example 3.13:- Given Ӽ = $\{s_1, s_2\}$, and the fuzzy subset of Ӽ is A = $\{(s_1, 0.3), (s_2, 0.3)$. Suppose $T_f = \{1_x, 0_x\}$ and $I_f = \{0_x\}$, then $T_f^* = \{1_x, 0_x\}$ and $\pi_f = \{1_x, 0_x\}$. Since A is fuzzy-preopen but not fuzzy-π-open.

Example 3.14:- Given Ӽ = $\{s_1, s_2\}$, and the fuzzy sets on Ӽ are A = $\{(s_1, 0.4), (s_2, 0.3)\}$ and B = $\{(s_1, 0.5), (s_2, 0.5)\}$. Suppose $T_f = \{1_x, 0_x, A\}$ and $I_f = \{0_x\}$, then $T_f^* = \{1_x, 0_x, A\}$ and $\pi_f = \{1_x, 0_x, A, B\}$.

Since int(cl(B)) = $\{(s_1, 0.4), (s_2, 0.3)\}$ thus B is not fuzzy-preopen but it is fuzzy-π-open.

Theorem 3.15:- If a fuzzy subset S of (Ӽ, T_f, I_f) space is fuzzy-preopen and fuzzy-semi-closed, then S is fuzzy-π-open.

Proof:- Suppose that S is fuzzy-preopen set, then $\mu_S(x) \leq \mu_{\mathrm{int(cl(S))}}(x)$ for every x in Ӽ, since S is fuzzy-semi-closed, therefore $\mu_{\mathrm{int(cl(S))}}(x) \leq \mu_S(x)$ for every x in Ӽ.

That is $\mu_{\text{int(cl(S))}}$(x) $\leq$ μ_{S}(x) $\leq$ $\mu_{\text{int(cl(S))}}$(x) for every x in Ҳ, thus int(cl(S) $\subseteq$ S $\subseteq$ int(cl(S)). Therefore

S = int(cl(S), that is S is fuzzy-regular-open, therefore it is fuzzy-open. Hence S is fuzzy-π-open by **[Corollary 3.10].**

<u>Theorem 3.16</u>:- In fuzzy ideal topological space (Ҳ, T_f, I_f) every fuzzy-semi-open sets is fuzzy-π-open but not contrary.

<u>Proof</u>:- Suppose S is fuzzy-semi-open set in (Ҳ, T_f, I_f) space, therefore μ_S(x) $\leq$ $\mu_{\text{cl(int(S))}}$(x) for every x in Ҳ. Since int(S) is fuzzy-open implies that it is fuzzy-T*-open and hence it is fuzzy-π-open, i.e int(S) $\subseteq$ int(S) $\cap$ ψ_{Tf}(int(S)). Thus

$\mu_{\text{int(S)}}$(x) $\leq$ min {$\mu_{\text{int(S)}}$(x), $\mu_{\psi_{Tf}(\text{int(S)})}$(x)}. Now int(S) $\subseteq$ S, therefore ψ_{Tf}(int(S))$\subseteq$ ψ_{Tf}(S) by [**Lemma 3.4**], since $\mu_{\text{int(S)}}$(x) $\leq$ min {$\mu_{\text{int(S)}}$(x), $\mu_{\psi_{Tf}(\text{int(S)})}$(x)}. Therefore int(S) $\subseteq$ S $\cap$ ψ_{Tf}(S). Since int(S) is fuzzy-T*-open and. Hence S is fuzzy-π-open set.

<u>Example 3.17</u>:- Given Ҳ = {s_1, s_2}, and the fuzzy sets on Ҳ are A = {(s_1, 0.8), (s_2, 0.7)} and B = {(s_1, 0.5), (s_2, 0.5)}. Suppose T_f = {1_x, 0_x, A} and I_f = {0_x, $B_{0.5}^{s1}$, $B_{0.5}^{s2}$, B} where $B_{0.5}^{s1}$ and $B_{0.5}^{s2}$ are fuzzy point s_1 and s_2 in the fuzzy set B, then T_f^*= {1_x, 0_x, A, $B_{0.5}^{s1}$, $B_{0.5}^{s2}$ B} and π_f = {1_x, 0_x, A, $B_{0.5}^{s1}$, $B_{0.5}^{s2}$, B}. Since cl(int(B)) = 0_x, thus B is fuzzy-π-open but not fuzzy-semi-open.

<u>Theorem 3.18</u>:- If a fuzzy subset S of (Ҳ, T_f, I_f) space is fuzzy-semi-open and fuzzy-pre-closed, then S is fuzzy-π-clopen.

<u>Proof</u>:- Suppose that S is fuzzy-semi-open set, then μ_S(x) $\leq$ $\mu_{\text{cl(int(S))}}$(x) for every x in Ҳ, since S is fuzzy-pre-closed, therefore $\mu_{\text{cl(int(S))}}$(x) $\leq$ μ_S(x) for every x in Ҳ.

That is $\mu_{\text{cl(int(S))}}$(x) $\leq$ μ_S(x) $\leq$ $\mu_{\text{cl(int(S))}}$(x). Therefore cl(int(S)) $\subseteq$ S $\subseteq$ cl(int(S)), hence

S = cl(int(S)), as cl(int(S)) is fuzzy-closed set and every fuzzy-closed is fuzzy-π-closed. Hence S is fuzzy-π-closed. Since every fuzzy-semi-open is fuzzy-π-open by **[Theorem 3.16]**, therefore S is fuzzy-π-clopen.

<u>Theorem 3.19</u>:- In fuzzy ideal topological space (Ҳ, T_f, I_f) every fuzzy-alpha-open sets is fuzzy-π-open but not contrary.

<u>Proof</u>:- Suppose S is a fuzzy-alpha-open set in (Ҳ, T_f, I_f) space, therefore S $\subseteq$ int(cl(int(S))). Since int(S) is fuzzy open and hence it is fuzzy-T*-open, thus int(S) is fuzzy-π-open by **[Theorem 3.8],** i.e int(S) $\subseteq$ int(S) $\cap$ ψ_{Tf}(int(S)). Now int(S) $\subseteq$ S, therefore ψ_{Tf}(int(S)) $\subseteq$ ψ_{Tf}(S) by **[Lemma 3.4]**, since int(S) $\subseteq$ int(S) $\cap$ ψ_{Tf}(int(S)). That is int(S) $\subseteq$ S $\cap$ ψ_{Tf}(S). Hence S is fuzzy-π-open.

<u>Example 3.20</u>:- Given Ҳ = {s_1, s_2}, and the fuzzy subsets of Ҳ are A = {(s_1, 0.8), (s_2, 0.7)} and B = {(s_1, 0.5), (s_2, 0.5)}. Suppose T_f = {1_x, 0_x, A} and I_f = {0_x, $B_{0.5}^{s1}$, $B_{0.5}^{s2}$, B}, then T_f^*= {1_x, 0_x, A, $B_{0.5}^{s1}$, $B_{0.5}^{s2}$, B} and π_f = {1_x, 0_x, A, $B_{0.5}^{s1}$, $B_{0.5}^{s2}$, B}. Since int(cl(int(B)) = 0_x, thus B is fuzzy-π-open but not fuzz-α-open.

Diagram (1) explain the relationship among fuzzy-π-open and fuzzy-open, fuzzy-α-open, fuzzy-T*-open, fuzzy-semi-open and fuzzy-preopen sets.

$$\begin{array}{ccccc} \text{Fuzzy-open} & \rightarrow & \text{Fuzzy-Semi-open} & \leftarrow & \text{Fuzzy-}\alpha\text{-open} \\ & & \downarrow & & \\ \text{Fuzzy-Preopen} & \rightleftharpoons & \text{Fuzzy-}\pi\text{-open} & \leftarrow & \text{Fuzzy-T}^*\text{-open} \end{array}$$

(1)

4. Fuzzy-π-Continuous Function

Definition 4.1:- A function f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) is said to be fuzzy-π_i-continuous briefly [f-π_i-con] iff the inverse image of every fuzzy-open set in Y is fuzzy-π-open in Ҳ.

Definition 4.2:- A function f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) is said to be fuzzy-π_{ii}-continuous briefly [f-π_{ii}-con] iff the inverse image of every fuzzy-π-open set in Y is fuzzy-π-open in Ҳ.

Remark 4.3:- Every fuzzy-π_{ii}-continuous function is fuzzy-π_i-continuous but not conversely.

Proof:- Suppose that f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) is f-π_{ii}-con and let V be a fuzzy-open set in Y, then V is fuzzy-π-open by **[Corollary 3.10]**, since f is f-π_{ii}-con therefore $f^{-1}(V)$ is fuzzy-π-open set in Ҳ. Hence f is f-π_i-con.

Example 4.4:- Let Ҳ = {a, b} and Y = {x, y}, the fuzzy subsets of Ҳ are A = {(a, 0.5), (b, 0.3)}, and B = {(a, 0.2), (b, 0.3}.

The fuzzy subsets on Y are D = {(x, 0.3), (y, 0.5)} and E = {(x, 0.8), (y, 0.6)}.We define f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) as f(a) = y and f(b) = x,

when T_f = {0_x, 1_x, A}, I_f = {0_x, $B^a_{0.2}$, $B^b_{0.3}$, B} and σ_f = {0_y, 1_y, D}, J_f = {0_y, $E^x_{0.8}$, $E^y_{0.6}$ E}.

Then π_f(Ҳ) = {0_x, 1_x, A, $B^a_{0.8}$, $B^b_{0.7}$, B^c} and $\pi_f(Y)$ = {0_y, 1_y, D, $E^x_{0.2}$, $E^y_{0.4}$, E^c}.

Since f is f-π_i-con, but $f^{-1}(E^c)$ = {(x, 0.4), (y, 0.2)} $\notin \pi_f$(Ҳ). Therefore f is not f-π_{ii}-con.

Theorem 4.5 :- Let f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) be f-π_{ii}-con and g: (Y, σ_f, J_f) → (Z, ρ_f, K_f) be fuzzy-π_i-continuous, then (g o f) is f-π_i-con function

Proof:- Let V be fuzzy open set in Z we must prove that $(g \circ f)^{-1}(V)$ is fuzzy-π-open in Ҳ. Since $(g \circ f)^{-1}(V) = (f^{-1} \circ g^{-1})(V) = f^{-1}(g^{-1}(V))$ as g is f-π_i-con, then $g^{-1}(V)$ is fuzzy-π-open in Y. Since f is f-π_{ii}-con, therefore $f^{-1}(g^{-1}(V))$ is fuzzy-π-open in Ҳ. Hence g ∘ f is f-π_i-con.

Theorem 4.6:- If f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) be a fuzzy function, then if

1) f is fuzzy-continuous ⇒ f is f-π_i-con.
2) f is fuzzy-semi-continuous ⇒ f is f-π_i-con.
3) f is fuzzy-alpha-continuous ⇒ f is f-π_i-con.
4) f is fuzzy-regular-continuous ⇒ f is f-π_i-con.

Proof:- 1) Suppose that f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) is fuzzy-continuous and let V be a fuzzy-open set in Y, therefore $f^{-1}(V)$ is fuzzy open, that is $f^{-1}(V)$ is fuzzy-π-open in Ҳ by **[Corollary 3.10]**. Hence f is f-π_i-con function.

2) Suppose that f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) is fuzzy-semi-continuous and let V be a fuzzy-open set in Y, therefore $f^{-1}(V)$ is fuzzy-semi open, that is $f^{-1}(V)$ is fuzzy-π-open in Ҳ by **[Theorem 3.16]**. Hence f is f-π_i-con function.

3) Suppose that f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) is fuzzy-alpha-continuous and let V be a fuzzy-open set in Y, therefore $f^{-1}(V)$ is fuzzy-alpha-open, thus $f^{-1}(V)$ is fuzzy-π-open set in Ҳ by **[Theorem 3.19]**. Hence f is f-π_i-con function.

4) Suppose that f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) is fuzzy-regular-continuous. Since every fuzzy-regular-open is fuzzy-open. Therefore f is f-π_i-con function by (1).

Diagram (2) explain the relationships among fuzzy-π_i-continuous, fuzzy-π_{ii}-continuous and fuzzy- continuous, fuzzy-α-continuous, fuzzy-semi-continuous, fuzzy-pre-continuous.

$$
\begin{array}{ccccc}
\text{Fuzzy-Semi- continuous} & \longleftarrow & \text{Fuzzy-}\alpha\text{-continuous} & \longleftarrow & \text{Fuzzy-continuous} \\
 & & \downarrow & & \\
\text{Fuzzy-Pre- continuous} & \not\rightleftarrows & \text{fuzzy-}\pi_i\text{-continuous} & \longleftarrow & \text{fuzzy-}\pi_{ii}\text{-continuous}
\end{array}
$$

(2)

Theorem 4.7:- The function f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) is fuzzy-π_i-continuous iff the inverse image of every fuzzy-closed in Y is fuzzy-π-closed in Ҳ.

Proof:- Suppose f is f-π_i-con function and F be fuzzy-closed set in Y. Thus 1_y - F is fuzzy-open in Y, since f is f-π_i-con, then $f^{-1}(1_y - F)$ is fuzzy-π-open set in Ҳ. That is $1_x - f^{-1}(F)$ is fuzzy-π-open set in Ҳ. Hence $f^{-1}(F)$ is fuzzy-π-closed set in Ҳ. That is the inverse image of every fuzzy-closed set in Y is fuzzy-π-closed in Ҳ.

Conversely;- Suppose the inverse image of every fuzzy-closed set in Y be fuzzy-closed set in Ҳ. Let G be fuzzy-open in Y. Then (1_y - G) is fuzzy-closed set in Y. That is $f^{-1}(1_y - G)$ is fuzzy-π-closed set in Ҳ. Thus $1_x - f^{-1}(G)$ is fuzzy-π-closed in Ҳ. Hence $f^{-1}(G)$ is fuzzy-π-open set in Ҳ. That is the inverse image of every fuzzy-open set in Y is fuzzy-π-open set in Ҳ. Hence f is f-π_i-con on Ҳ.

Theorem 4.8:- Let f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) be a fuzzy function, then f is f-π_i-con if $f^{-1}(F) \in I_f$ for every fuzzy-closed set F in Y

Proof:- Suppose F be a fuzzy-closed set in Y and $f^{-1}(F) \in I_f$, then $f^{-1}(F)$ is fuzzy-T^*-closed. Thus $f^{-1}(F)$ fuzzy-π-closed, therefore f is f-π_i-con by **[Theorem 4.7].**

Definition 4.9:- The function f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) is called fuzzy-π_{ii}-open function if the image of every fuzzy-π-open set A in Ҳ is fuzzy-π-open in Y.

Definition 4.10:- The function f: (Ҳ, T_f, I_f) → (Y, σ_f, J_f) is called fuzzy-π_{ii}-closed function if the image of every fuzzy-π-closed set A in Ҳ is fuzzy-π-closed in Y.

Definition 4.11:- The bijective function f: (Ҳ, π_{1f}) → (Y, π_{2f}) is called fuzzy-π_{ii}-homeomorphism if f is f-π_{ii}-con and fuzzy-π_{ii}-open. And we said that (Ҳ, π_{1f}) space is homeomorphic to a space (Y, π_{2f}) and symbolized by Ҳ $\cong$ Y.

A property P of a fuzzy spaces is called a fuzzy topological property if whenever the fuzzy topological space (Ҳ, π_{1f}) has P then every homeomorphic fuzzy topological space to (Ҳ, π_{1f}) also has P. A property T of a fuzzy topological space is called an associated property if whenever the fuzzy topological space (Ҳ, π_{1f}) has T then the associated ordinary topology π_{2f} has T also.

5. Conclusion:

During our study of this space, and based on the results we obtained from our definition of fuzzy-π-open sets we concluded that the fuzzy ideal topological spaces is a vast domain that can be further explored. Most of researches conducted in this field relies on three types of sets, the fuzzy-open, fuzzy-T^*-open and the ideal sets. The majority of definitions are interlinked among these sets and established concepts. Moreover any alteration in the topological space or the ideal yields significantly different outcomes.Therefore we recommend an expansion in the study of this space.

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